

# A SHARP BILINEAR ESTIMATE FOR THE KLEIN–GORDON EQUATION IN ARBITRARY SPACE-TIME DIMENSIONS

CHRIS JEAVONS

**ABSTRACT.** We prove a sharp bilinear inequality for the Klein–Gordon equation on  $\mathbb{R}^{d+1}$ , for any  $d \geq 2$ . This extends work of Ozawa–Rogers and Quilodr  n for the Klein–Gordon equation and generalises work of Bez–Rogers for the wave equation. As a consequence we obtain a sharp Strichartz estimate for the solution of the Klein–Gordon equation in five spatial dimensions for data belonging to  $H^1$ . We show that maximisers for this estimate do not exist and that any maximising sequence of initial data concentrates at spatial infinity.

## 1. INTRODUCTION

For the Klein–Gordon equation on  $\mathbb{R}^{1+1}$ , very recently in [20] it was shown that the bilinear estimate

$$(1.1) \quad \left\| e^{it\sqrt{1-\Delta}} f_1 \, e^{it\sqrt{1-\Delta}} f_2 \right\|_{L^2(\mathbb{R}^2)}^2 \leq \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} |\widehat{f}_1(y_1)|^2 |\widehat{f}_2(y_2)|^2 \frac{(1+y_1^2)^{\frac{3}{4}} (1+y_2^2)^{\frac{3}{4}}}{|y_1 - y_2|} dy_1 dy_2$$

holds whenever  $f_1$  and  $f_2$  have disjoint Fourier supports, and that the constant  $\frac{1}{(2\pi)^2}$  is sharp. The main motivation behind the present paper was to identify a natural generalisation of this sharp bilinear estimate to arbitrary dimensions. In achieving this, we simultaneously extend work of Quilodr  n in [21] and generalise work of Bez–Rogers [2]. We will also obtain a new Strichartz estimate with sharp constant for the Klein–Gordon equation on  $\mathbb{R}^{5+1}$  with  $H^1$ -initial data.

Throughout this paper, we let  $\widehat{\cdot}$  denote the spatial Fourier transform on  $\mathbb{R}^d$ , defined on the Schwartz class as

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-ix \cdot \xi} dx, \quad \xi \in \mathbb{R}^d.$$

For fixed  $s \geq 0$ , define also the Klein–Gordon propagator  $e^{it\phi_s(\sqrt{-\Delta})}$  by

$$(1.2) \quad e^{it\phi_s(\sqrt{-\Delta})} f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{f}(\xi) e^{ix \cdot \xi + it(s^2 + |\xi|^2)^{\frac{1}{2}}} d\xi, \quad x \in \mathbb{R}^d, \quad t \in \mathbb{R},$$

on the Schwartz class, where  $\phi_s(r) = \sqrt{s^2 + r^2}$ , for  $r \in \mathbb{R}$ . In order to state the main result, in what follows we let

$$K_s(y_1, y_2) = \frac{(\phi_s(|y_1|)\phi_s(|y_2|) - y_1 \cdot y_2 - s^2)^{\frac{d-2}{2}}}{(\phi_s(|y_1|)\phi_s(|y_2|) - y_1 \cdot y_2 + s^2)^{\frac{1}{2}}}$$

be a function on  $\mathbb{R}^{2d}$ , and we introduce the constant

$$\mathbf{KG}(d) = \frac{2^{-\frac{d-1}{2}} |\mathbb{S}^{d-1}|}{(2\pi)^{3d-1}}$$

for  $d \geq 1$ , which will appear throughout the paper.

**Theorem 1.** *If  $d \geq 2$  and  $s \geq 0$ , then*

$$(1.3) \quad \left\| e^{it\phi_s(\sqrt{-\Delta})} f_1 e^{it\phi_s(\sqrt{-\Delta})} f_2 \right\|_{L^2(\mathbb{R}^{d+1})}^2 \leq \mathbf{KG}(d) \int_{\mathbb{R}^{2d}} |\widehat{f}_1(y_1)|^2 |\widehat{f}_2(y_2)|^2 \phi_s(|y_1|) \phi_s(|y_2|) K_s(y_1, y_2) dy_1 dy_2,$$

where the constant  $\mathbf{KG}(d)$  is best possible since we have equality for functions of the form

$$(1.4) \quad \widehat{f}_1(\xi) = \widehat{f}_2(\xi) = \frac{e^{-a\phi_s(|\xi|)}}{\phi_s(|\xi|)},$$

for  $a > 0$ . Further, if  $d = 1$ ,  $s > 0$  and  $\widehat{f}_1, \widehat{f}_2$  have disjoint support, or  $s = 0$  and  $\widehat{f}_1, \widehat{f}_2$  have disjoint angular support, then we have

$$(1.5) \quad \left\| e^{it\phi_s(\sqrt{-\Delta})} f_1 e^{it\phi_s(\sqrt{-\Delta})} f_2 \right\|_{L^2(\mathbb{R}^{1+1})}^2 = \frac{\mathbf{KG}(1)}{2} \int_{\mathbb{R}^{2d}} |\widehat{f}_1(y_1)|^2 |\widehat{f}_2(y_2)|^2 \phi_s(|y_1|) \phi_s(|y_2|) K_s(y_1, y_2) dy_1 dy_2.$$

It turns out that the functions described in (1.4) above play an important role in some of the applications of this inequality, as we will see below in Corollary 2.

In the case  $d = 1$ , we observe that

$$(1.6) \quad K_s(y_1, y_2) \leq \frac{1}{s^2} \frac{(s^2 + y_1^2)^{\frac{1}{4}} (s^2 + y_2^2)^{\frac{1}{4}}}{|y_1 - y_2|}$$

for almost every  $(y_1, y_2) \in \mathbb{R}^2$  with  $y_1 \neq y_2$  and  $s > 0$ . One can see this by first reducing to the case  $s = 1$  and then a direct argument shows that the claimed inequality is equivalent to

$$0 \leq (y_1^2 + y_2^2)(y_1 - y_2)^4 + y_1^2 y_2^2 (y_1 - y_2)^4,$$

which is clearly true. Since  $\frac{1}{2}\mathbf{KG}(1) = \frac{1}{(2\pi)^2}$ , we see that (1.1) follows from (1.5)<sup>1</sup> and we claim that (1.3) in Theorem 1 provides a natural generalisation of this to higher dimensions.

Furthermore, one can deduce certain Strichartz estimates from (1.3) with sharp constants, some of which recover sharp Strichartz estimates due to Quilodrán in [21] and Bez–Rogers in [2], and we also obtain a new sharp Strichartz estimate for the Klein–Gordon equation in five spatial dimensions (see the forthcoming Corollary 2). In order to describe these results, we define the inhomogeneous Sobolev norm as follows

$$\|f\|_{H^m(\mathbb{R}^d)} = \|(1 + |\cdot|^2)^{\frac{m}{2}} \widehat{f}\|_{L^2(\mathbb{R}^d)}, \quad m \geq 0.$$

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<sup>1</sup>in fact, the argument in [20] leading to (1.1) goes via the identity (1.5), and they prove (1.6) differently using some trigonometric identities

When  $d = 2$  and  $s = 1$ , we have

$$(1.7) \quad K_s(y_1, y_2) = (\phi_1(|y_1|)\phi_1(|y_2|) - y_1 \cdot y_2 + 1)^{-\frac{1}{2}} \leq 2^{-\frac{1}{2}},$$

for  $(y_1, y_2) \in \mathbb{R}^4$ . Taking  $f_1 = f_2$  in (1.3) it follows that

$$(1.8) \quad \|e^{it\sqrt{1-\Delta}}f\|_{L^4(\mathbb{R}^{2+1})} \leq \frac{1}{2^{5/4}\pi} \|f\|_{H^{1/2}(\mathbb{R}^2)}.$$

Estimate (1.8) is due to Quilodr  n [21] and he showed that the constant is sharp but that maximisers do not exist<sup>2</sup>.

Similarly, when  $d = 3$  and  $s = 1$ , we get

$$(1.9) \quad K_s(y_1, y_2) = \frac{(\phi_1(|y_1|)\phi_1(|y_2|) - y_1 \cdot y_2 - 1)^{\frac{1}{2}}}{(\phi_1(|y_1|)\phi_1(|y_2|) - y_1 \cdot y_2 + 1)^{\frac{1}{2}}} \leq 1,$$

and (1.3) implies

$$(1.10) \quad \|e^{it\sqrt{1-\Delta}}f\|_{L^4(\mathbb{R}^{3+1})} \leq \frac{1}{(2\pi)^{7/4}} \|f\|_{H^{1/2}(\mathbb{R}^3)}.$$

Again, the constant is sharp and maximisers do not exist (due to Quilodr  n [21]). We remark that we prove Theorem 1 using the approach of Foschi in [13], as did Quilodr  n, and so it is not at all a surprise that (1.8) and (1.10) follow from Theorem 1.

In this paper, we obtain the following new sharp form of a classical Strichartz estimate for the full solution of the Klein–Gordon equation for data in the energy space.

**Corollary 1.** *Suppose that  $\partial_{tt}u - \Delta u + u = 0$  on  $\mathbb{R}^{5+1}$ , then*

$$(1.11) \quad \|u\|_{L^4(\mathbb{R}^{5+1})} \leq \left(\frac{1}{8\pi}\right)^{\frac{1}{2}} \left(\|u(0)\|_{H^1(\mathbb{R}^5)}^2 + \|\partial_t u(0)\|_{L^2(\mathbb{R}^5)}^2\right)^{\frac{1}{2}}.$$

*The constant  $(\frac{1}{8\pi})^{\frac{1}{2}}$  is sharp, but there are no nontrivial functions for which we have equality.*

A nonsharp form of (1.11) was proved by Strichartz in [26]. The sharp inequality (1.11) is deduced from the following sharp estimate for the one-sided propagator  $e^{it\phi_s(\sqrt{-\Delta})}$ . In order to state this result, we introduce the notation

$$\|f\|_{(s)}^4 := \left\| \phi_s(\sqrt{-\Delta})f \right\|_{L^2}^4 - s^2 \left\| (\phi_s(\sqrt{-\Delta}))^{\frac{1}{2}} f \right\|_{L^2}^4.$$

Notice that if  $s = 1$  then  $\|f\|_{(s)}$  may be bounded above by the inhomogeneous norm  $\|\cdot\|_{H^1}$ , and if  $s = 0$  then  $\|f\|_{(s)}$  is just the  $\dot{H}^1$ -norm of  $f$ .

**Corollary 2.** *Let  $s \geq 0$ . Then*

$$(1.12) \quad \left\| e^{it\phi_s(\sqrt{-\Delta})}f \right\|_{L^4(\mathbb{R}^{5+1})} \leq \left(\frac{1}{24\pi^2}\right)^{\frac{1}{4}} \|f\|_{(s)}.$$

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<sup>2</sup>in [21] the perspective is that of adjoint Fourier restriction inequalities for the hyperboloid, and we choose to present the estimates from (1.8) in terms of the Klein–Gordon propagator

The constant  $\left(\frac{1}{24\pi^2}\right)^{\frac{1}{4}}$  is sharp as we have the maximising sequence  $(g_a)_{a>0}$  defined by

$$(1.13) \quad g_a = \frac{f_a}{\|f_a\|_{(s)}},$$

where

$$(1.14) \quad \widehat{f}_a(\xi) = \frac{e^{-a\phi_s(|\xi|)}}{\phi_s(|\xi|)}$$

as  $a \rightarrow 0+$ , but when  $s > 0$  there are no functions for which we have equality.

By maximising sequence for (1.12) we mean a sequence of functions  $(g_n)_{n \geq 1}$  satisfying  $\|g_n\|_{(s)} \leq 1$  for which

$$\left\| e^{it\phi_s(\sqrt{-\Delta})} g_n \right\|_{L^4(\mathbb{R}^{5+1})} \rightarrow \left( \frac{1}{24\pi^2} \right)^{\frac{1}{4}}$$

as  $n \rightarrow \infty$ .

When  $s = 1$ , (1.12) is the sharp estimate

$$(1.15) \quad \|e^{it\sqrt{1-\Delta}} f\|_{L^4(\mathbb{R}^{5+1})} \leq \left( \frac{1}{24\pi^2} \right)^{\frac{1}{4}} (\|f\|_{H^1(\mathbb{R}^5)}^4 - \|f\|_{H^{1/2}(\mathbb{R}^5)}^4)^{1/4},$$

which is a refinement of the sharp estimate

$$(1.16) \quad \|e^{it\sqrt{1-\Delta}} f\|_{L^4(\mathbb{R}^{5+1})} \leq \left( \frac{1}{24\pi^2} \right)^{\frac{1}{4}} \|f\|_{H^1(\mathbb{R}^5)}.$$

Both (1.15) and (1.16) are new. With nonsharp constant, (1.16) follows from [26]. That the constant in (1.16) is sharp follows from the observation that for the functions  $f_a$  defined by (1.14) one has that

$$(1.17) \quad a^5 \left\| \phi_s(\sqrt{-\Delta})^{\frac{1}{2}} f_a \right\|_{L^2(\mathbb{R}^5)}^2 \rightarrow 0$$

as  $a \rightarrow 0+$  (see Section 3). In fact, a similar property holds for maximising sequences for (1.12), as we will see in our forthcoming Proposition 1.

At this point, we make some remarks concerning the particular case of the wave equation corresponding to the case  $s = 0$ . With the emphasis not on sharp constants, Klainerman–Machedon first established bilinear estimates in the spirit of (1.3) with different kinds of weights in the case  $s = 0$  (see [16], [17], [18]). Regarding sharp estimates, Theorem 1 and (1.12) for  $s = 0$  were established in [2] (see also [6] for similar results for the Schrödinger propagator). Also, maximisers exist in both cases  $s = 0$  and  $s > 0$  in Theorem 1. However, in Corollary 2, it is true that when  $s = 0$  and for any  $a > 0$ , the function  $f_a$  given by (1.14) is a maximiser, but when  $s > 0$ , there are no maximisers and when suitably normalised, the functions  $f_a$  form a maximising sequence as  $a$  tends to zero.

As our final main result in this paper, we establish that *any* maximising sequence for the estimate (1.12) must concentrate at spatial infinity in the following precise sense.

**Proposition 1.** *If  $(g_n)_{n \geq 1}$  is any maximising sequence for (1.12), then for each  $\varepsilon, R > 0$  there exists  $N \in \mathbb{N}$  so that if  $n \geq N$ ,*

$$\left\| \phi_s(\sqrt{-\Delta})^{\frac{1}{2}} g_n \right\|_{L^2(\mathbb{R}^5)} < \varepsilon,$$

and

$$(1.18) \quad \left\| \widehat{\phi_s(\sqrt{-\Delta}) g_n} \right\|_{L^2(B(0,R))} < \varepsilon,$$

where  $B(0, R)$  denotes the ball of radius  $R$  centered at the origin in  $\mathbb{R}^5$ .

The motivation for this result comes from the observation that the particular maximising sequence  $(g_a)$  considered in Corollary 2 satisfies these conditions. A result analogous to (1.18) was established in [21], where it was shown that any maximising sequence for either (1.8) or (1.10) must concentrate at spatial infinity. We also remark here that in the case  $s = 1$ , Proposition 1 may be interpreted as a statement about the concentration of the  $H^1$ -norm of a maximising sequence for the inequality (1.12).

Largely as a result of the influential work of Foschi [13], a body of very recent work has emerged on sharp constants and the existence or nature of maximisers for space-time estimates associated with dispersive PDE, to which this work belongs. In addition to [2], [13], [20] and [21] already mentioned, see for example, [1], [3], [6], [9], [15], [25], and [27] for sharp constants, and [4], [7], [8], [10], [11], [19], [22], and [24] for results on maximisers.

## 2. PROOF OF THEOREM 1

**2.1. The case  $d \geq 2$  and  $s \geq 0$ .** In this section we will use the space-time Fourier transform, defined for suitable functions  $f$  on  $\mathbb{R}^{d+1}$  by

$$\tilde{f}(\xi, \tau) = \int_{\mathbb{R}^{d+1}} f(x, t) e^{-i(t\tau + x \cdot \xi)} dx dt.$$

We note firstly that the space-time Fourier transform of  $v_j = e^{it\phi_s(\sqrt{-\Delta})} f_j$  will be the measure

$$\tilde{v}_j(\xi, \tau) = 2\pi \delta(\tau - \phi_s(|\xi|)) \widehat{f_j}(\xi)$$

for  $j = 1, 2$ , each supported on the hyperboloid in  $\mathbb{R}^{d+1}$ ,

$$\left\{ (y, (s^2 + |y|^2)^{\frac{1}{2}}) : y \in \mathbb{R}^d \right\}.$$

Note that if  $s > 0$  and  $d \geq 2$ , the function defined by  $K_s$  is well-defined for any  $y = (y_1, y_2) \in \mathbb{R}^{2d}$ . For example, if  $d = 2$  the kernel reduces to

$$K_s(y_1, y_2) = \frac{1}{(s^2 + \phi_s(|y_1|)\phi_s(|y_2|) - y_1 \cdot y_2)^{\frac{1}{2}}},$$

and the denominator is always positive since

$$y_1 \cdot y_2 \leq |y_1| |y_2| < (|y_1|^2 + s^2)^{\frac{1}{2}} (|y_2|^2 + s^2)^{\frac{1}{2}} + s^2 = \phi_s(|y_1|)\phi_s(|y_2|) + s^2;$$

the claim for  $d > 2$  follows from this as the power  $\frac{d-2}{2}$  is positive in this case.

If we now write  $u_2 = e^{it\phi_s(\sqrt{-\Delta})} f_1 e^{it\phi_s(\sqrt{-\Delta})} f_2$ , then the space-time Fourier transform of  $u_2$  will be the convolution of the measures  $\tilde{v}_1$  and  $\tilde{v}_2$ , which may be written as

$$(2.1) \quad \widehat{u_2}(\xi, \tau) = \frac{1}{(2\pi)^{d-1}} \int_{\mathbb{R}^{2d}} \frac{\widehat{F}(y)}{(s^2 + |y_1|^2)^{\frac{1}{4}} (s^2 + |y_2|^2)^{\frac{1}{4}}} \delta \left( \begin{array}{c} \tau - \phi_s(|y_1|) - \phi_s(|y_2|) \\ \xi - y_1 - y_2 \end{array} \right) dy,$$

where  $\xi \in \mathbb{R}^d$  and  $\tau \in \mathbb{R}$  are fixed, we set

$$\widehat{F}(y) = \widehat{f_1}(y_1) \widehat{f_2}(y_2) (s^2 + |y_1|^2)^{\frac{1}{4}} (s^2 + |y_2|^2)^{\frac{1}{4}},$$

and we use the notation  $\delta \left( \begin{smallmatrix} t \\ x \end{smallmatrix} \right)$  for the product  $\delta(t)\delta(x)$  on  $\mathbb{R}^{d+1}$ . It is proved in [21] that the function  $\widehat{u_2}$  is supported on the set

$$\mathcal{H}_s = \left\{ (\xi, \tau) \in \mathbb{R}^{d+1} : \tau \geq ((2s)^2 + |\xi|^2)^{\frac{1}{2}} \right\},$$

for completeness we include the proof here. If  $\xi = y_1 + y_2$  and  $\tau = \phi_s(|y_1|) + \phi_s(|y_2|)$  we have that

$$\begin{aligned} \tau^2 &= 2s^2 + |y_1|^2 + |y_2|^2 + 2\phi_s(|y_1|)\phi_s(|y_2|) \\ &\geq |y_1|^2 + |y_2|^2 + 2|y_1||y_2| + 4s^2 \\ &\geq 4s^2 + |\xi|^2, \end{aligned}$$

since

$$(2.2) \quad \phi_s(|y_1|)\phi_s(|y_2|) \geq s^2 + |y_1||y_2|,$$

as can easily be seen by squaring both sides. By the Cauchy–Schwarz inequality,

$$(2.3) \quad |\widehat{u_2}(\xi, \tau)|^2 \leq \frac{I_s(\xi, \tau)}{(2\pi)^{2d-2}} \int_{\mathbb{R}^{2d}} |\widehat{F}(y)|^2 K_s(y) \delta \left( \begin{array}{c} \tau - \phi_s(|y_1|) - \phi_s(|y_2|) \\ \xi - y_1 - y_2 \end{array} \right) dy,$$

where

$$I_s(\xi, \tau) = \int_{\mathbb{R}^{2d}} \frac{1}{K_s(y) \phi_s(|y_1|) \phi_s(|y_2|)} \delta \left( \begin{array}{c} \tau - \phi_s(|y_1|) - \phi_s(|y_2|) \\ \xi - y_1 - y_2 \end{array} \right) dy.$$

Now, on the support of the delta measures, by the choice of  $K_s$  we have that

$$K_s(y) = \frac{1}{2^{\frac{d-3}{2}}} \frac{(\tau^2 - |\xi|^2 - 4s^2)^{\frac{d-2}{2}}}{(\tau^2 - |\xi|^2)^{\frac{1}{2}}},$$

so that

$$\begin{aligned} I_s(\xi, \tau) &= \frac{2^{\frac{d-3}{2}} (\tau^2 - |\xi|^2)^{\frac{1}{2}}}{(\tau^2 - |\xi|^2 - 4s^2)^{\frac{d-2}{2}}} \int_{\mathbb{R}^{2d}} \frac{1}{\phi_s(|y_1|) \phi_s(|y_2|)} \delta \left( \begin{array}{c} \tau - \phi_s(|y_1|) - \phi_s(|y_2|) \\ \xi - y_1 - y_2 \end{array} \right) dy \\ &= \frac{2^{\frac{d-3}{2}} (\tau^2 - |\xi|^2)^{\frac{1}{2}}}{(\tau^2 - |\xi|^2 - 4s^2)^{\frac{d-2}{2}}} \sigma_s * \sigma_s(\xi, \tau), \end{aligned}$$

where we have defined the measure  $\sigma_s$  on  $\mathbb{R}^{d+1}$  by

$$\int_{\mathbb{R}^{d+1}} g(x, t) d\sigma_s(x, t) = \int_{\mathbb{R}^{d+1}} g(x, t) \delta(t - \phi_s(|x|)) \frac{dx dt}{\phi_s(|x|)}.$$

Indeed, since  $\xi = y_1 + y_2$  we have that  $|\xi|^2 = |y_1|^2 + |y_2|^2 + 2y_1 \cdot y_2$ , and since  $\tau = \phi_s(|y_1|) + \phi_s(|y_2|)$  it follows that

$$\tau^2 = 2s^2 + |y_1|^2 + |y_2|^2 + 2(s^2 + |y_1|^2)^{\frac{1}{2}}(s^2 + |y_2|^2)^{\frac{1}{2}},$$

and so we obtain

$$\tau^2 - |\xi|^2 = 2(s^2 + |y_1|^2)^{\frac{1}{2}}(s^2 + |y_2|^2)^{\frac{1}{2}} + 2s^2 - 2y_1 \cdot y_2,$$

so that

$$\frac{1}{2}(\tau^2 - |\xi|^2) = \phi_s(|y_1|)\phi_s(|y_2|) + s^2 - y_1 \cdot y_2,$$

and

$$\phi_s(|y_1|)\phi_s(|y_2|) - s^2 - y_1 \cdot y_2 = \frac{1}{2}(\tau^2 - |\xi|^2 - 4s^2).$$

Hence we need to compute the quantity

$$J_s(\xi, \tau) := \int_{\mathbb{R}^{2d}} \frac{1}{\phi_s(|y_1|)\phi_s(|y_2|)} \delta\left(\begin{matrix} \tau - \phi_s(|y_1|) - \phi_s(|y_2|) \\ \xi - y_1 - y_2 \end{matrix}\right) dy = \sigma_s * \sigma_s(\xi, \tau).$$

It is known, [26], that the measure  $\sigma_s$  is invariant under Lorentz transformations, and hence so is the convolution  $J_s$ . Using this invariance, the convolution may be computed easily.

**Lemma 1.** *For all  $(\xi, \tau) \in \mathcal{H}_s$  we have that*

$$J_s(\xi, \tau) = \frac{|\mathbb{S}^{d-1}|}{2^{d-2}} \left(\tau^2 - |\xi|^2 - 4s^2\right)^{\frac{d-2}{2}} \left(\tau^2 - |\xi|^2\right)^{-\frac{1}{2}},$$

and hence

$$I_{2,s}(\xi, \tau) = \frac{|\mathbb{S}^{d-1}|}{2^{\frac{d-1}{2}}}.$$

*Proof.* As in [21] we use a one-parameter subgroup of transformations  $\{L^t\}_{t \in (-1,1)}$  of the group of Lorentz transformations from  $\mathbb{R}^{d+1}$  to itself, defined as

$$L^t(\xi, \tau) = \left( \frac{\xi_1 + t\tau}{\sqrt{1-t^2}}, \xi_2, \dots, \xi_d, \frac{\tau + t\xi_1}{\sqrt{1-t^2}} \right),$$

for  $(\xi, \tau) \in \mathbb{R}^{d+1}$ . We then note that the map  $(\xi, \tau) \rightarrow (A\xi, \tau)$  for any rotation  $A$  of  $\mathbb{R}^d$  also belongs to the group of Lorentz transformations, and so for fixed  $(\xi, \tau)$  if we compose the operator  $L^t$  where  $t = -\frac{|\xi|}{\tau}$  with the map described above satisfying  $A\xi = (|\xi|, 0, \dots, 0)$  we obtain a Lorentz transformation  $\mathcal{L}$  so that  $\mathcal{L}(\xi, \tau) = (0, (\tau^2 - |\xi|^2)^{\frac{1}{2}})$ . But then, as  $|\det \mathcal{L}| = 1$  it follows that the convolution  $\sigma_s * \sigma_s$  is also invariant under  $\mathcal{L}$ , and hence

$$J_s(\xi, \tau) = J_s(0, (\tau^2 - |\xi|^2)^{\frac{1}{2}}), \quad \tau > |\xi|.$$

This important reduction means that it suffices to consider  $J_s(0, z)$  for  $z \in \mathbb{R}$ . Now

$$\begin{aligned} J_s(0, z) &= \int_{\mathbb{R}^{d+1}} \delta(z - t - \phi_s(|-y|)) \frac{1}{\phi_s(|-y|)} \delta(t - \phi_s(|y|)) \frac{dy}{\phi_s(|y|)} dt \\ &= \int_{\mathbb{R}^d} \delta(z - 2\phi_s(|y|)) \frac{dy}{\phi_s(|y|)^2}. \end{aligned}$$

Using polar co-ordinates, we obtain

$$\sigma_s * \sigma_s(0, z) = |\mathbb{S}^{d-1}| \int_0^\infty \delta(z - 2\phi_s(r)) \frac{r^{d-1}}{\phi_s(r)^2} dr.$$

If we now make the change of variables  $u = 2\phi_s(r)$ , by the definition of  $\phi_s$  we have that  $\frac{r}{\phi_s(r)^2} dr = \frac{du}{u}$  and so

$$\begin{aligned}\sigma_s * \sigma_s(0, z) &= |\mathbb{S}^{d-1}| \int_{2s}^{\infty} \delta(z - u) \left( \frac{1}{2} \sqrt{u^2 - 4s^2} \right)^{d-2} \frac{du}{u} \\ &= \frac{|\mathbb{S}^{d-1}|}{2^{d-2}} \chi_{\{z \geq 2s\}} \frac{(z^2 - 4s^2)^{\frac{d-2}{2}}}{z},\end{aligned}$$

and the desired result follows from the Lorentz invariance discussed above.  $\square$

If we now integrate the inequality (2.3) for  $|\widehat{u}_2|^2$  with respect to  $\tau$  and  $\xi$ , apply Plancherel's theorem and change the order of integration, we obtain

$$\begin{aligned}\|u_2\|_{L_{x,t}^2}^2 &= \frac{1}{(2\pi)^{d+1}} \|\widehat{u}_2\|_{L_{\xi,\tau}^2}^2 \\ &\leq \frac{2^{-\frac{d-1}{2}} |\mathbb{S}^{d-1}|}{(2\pi)^{3d-1}} \int_{\mathbb{R}^{2d}} |\widehat{f}_1(y_1)|^2 |\widehat{f}_2(y_2)|^2 K_s(y) \phi_s(|y_1|) \phi_s(|y_2|) dy.\end{aligned}$$

Moreover, if we consider the functions  $f_j$  defined by

$$(2.4) \quad \phi_s(|y_j|) \widehat{f}_j(y_j) = e^{-a\phi_s(|y_j|)},$$

for  $a > 0$  (and  $j = 1, 2$ ), we immediately obtain that

$$\widehat{F}(y) = \frac{e^{-a\tau}}{\sqrt{\phi_s(|y_1|) \phi_s(|y_2|)}},$$

on the support of the delta measures. Since the only place an inequality was used was in the application of the Cauchy–Schwarz inequality, this implies that we have equality for such functions. Indeed, the above equality implies the existence of a scalar function  $g = g_s(\xi, \tau)$  so that

$$K_s(y) \widehat{F}(y) = g(\xi, \tau) K_s(y)^{-1} (\phi_s(|y_1|) \phi_s(|y_2|))^{-\frac{1}{2}}$$

almost everywhere on the support of the delta measures, since on this set  $K_s$  may be written in terms of  $\tau, \xi$  and  $s$  only, as shown above, and so may be absorbed into the function  $g$ . Hence we have equality in (2.3) for these functions  $f_j$ , and thus also in (1.3) for the constant

$$\mathbf{KG}(d) = \frac{2^{-\frac{d-1}{2}} |\mathbb{S}^{d-1}|}{(2\pi)^{3d-1}},$$

implying that it is best possible.

**2.2. The case  $d = 1$  and  $s > 0$ .** We note that formally, the calculation allowing us to derive (1.3) also makes sense for  $d = 1$ . However, substituting  $d = 1$  into the expression for  $K_s$  gives

$$\begin{aligned}K_s(y) &= [(\phi_s(|y_1|) \phi_s(|y_2|) - y_1 y_2 + s^2) (\phi_s(|y_1|) \phi_s(|y_2|) - y_1 y_2 - s^2)]^{-\frac{1}{2}} \\ &= \left( (\phi_s(|y_1|) \phi_s(|y_2|) - y_1 y_2)^2 - s^4 \right)^{-\frac{1}{2}} \\ &= (s^2(y_1^2 + y_2^2) + 2y_1^2 y_2^2 - 2y_1 y_2 \phi_s(|y_1|) \phi_s(|y_2|))^{-\frac{1}{2}},\end{aligned}$$

and since this weight is singular on the diagonal  $\{(y_1, y_2) \in \mathbb{R}^2 : y_1 = y_2\}$ , it is not difficult to construct a pair of integrable functions  $(f_1, f_2)$  for which the integral



given by the right hand side of (1.3) is unbounded. However if  $s > 0$  the weight  $K_s$  is well-defined for  $y_1 \neq y_2$  and if we assume that  $f_1$  and  $f_2$  have disjointly supported Fourier transforms, we have the identity (1.5). To prove (1.5) we follow a method used in [12] for restriction estimates on the sphere (see also [14], [23], [5], [20]). Specifically, we write

$$\begin{aligned} & e^{it\phi_s(\sqrt{-\Delta})} f_1(x) \overline{e^{it\phi_s(\sqrt{-\Delta})} f_2(x)} \\ &= \int_{\mathbb{R}^2} e^{ix(y_1-y_2)} e^{it((s^2+y_1^2)^{\frac{1}{2}}-(s^2+y_2^2)^{\frac{1}{2}})} \widehat{f_1}(y_1) \overline{\widehat{f_2}(y_2)} dy_1 dy_2. \end{aligned}$$

If we make the change of variables  $(y_1, y_2) \mapsto (u, v)$ , where  $u = y_1 - y_2$  and  $v = \phi_s(|y_1|) - \phi_s(|y_2|)$ , then the Jacobian will be

$$\left| \det \begin{pmatrix} \frac{1}{\sqrt{s^2+y_1^2}} & -\frac{1}{\sqrt{s^2+y_2^2}} \\ \frac{y_1}{\sqrt{s^2+y_1^2}} & -\frac{y_2}{\sqrt{s^2+y_2^2}} \end{pmatrix} \right|^{-1} = \frac{\phi_s(|y_1|)\phi_s(|y_2|)}{|\phi_s(|y_1|)y_2 - \phi_s(|y_2|)y_1|}.$$

Hence, we have

$$e^{it\phi_s(\sqrt{-\Delta})} f_1(x) \overline{e^{it\phi_s(\sqrt{-\Delta})} f_2(x)} = \int_{\mathbb{R}_+^2} e^{ixu} e^{itv} H(u, v) du dv,$$

where  $H$  is defined by

$$H(u, v) = \frac{\phi_s(|y_1|)\phi_s(|y_2|)}{|\phi_s(|y_1|)y_2 - \phi_s(|y_2|)y_1|} \widehat{f_1}(y_1) \widehat{f_2}(y_2).$$

By Plancherel's theorem,

$$\left\| e^{it\phi_s(\sqrt{-\Delta})} f_1 e^{it\phi_s(\sqrt{-\Delta})} f_2 \right\|_{L_{x,t}^2}^2 = \frac{1}{(2\pi)^2} \|H\|_{L_{u,v}^2}^2 = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} |H(u, v)|^2 du dv.$$

By reversing the change of variables done in the previous step, this becomes

$$\|u\|_{L^2}^2 = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} |\widehat{f_1}(y_1)|^2 |\widehat{f_2}(y_2)|^2 \frac{\phi_s(|y_1|)\phi_s(|y_2|)}{|y_1\phi_s(|y_2|) - y_2\phi_s(|y_1|)} dy_1 dy_2.$$

Further, by a direct calculation, it is easily verified that

$$\left( (s^2 + y_1^2)^{\frac{1}{2}} y_2 - (s^2 + y_2^2)^{\frac{1}{2}} y_1 \right)^2 = K_s(y)^{-2}.$$

Note that from the above we can see that the only singularity of the weight  $K_s$  would be at a point in  $\mathbb{R}^2$  where

$$y_1(s^2 + y_2^2)^{\frac{1}{2}} = y_2(s^2 + y_1^2)^{\frac{1}{2}},$$

which can only happen if  $y_1 = y_2$ . It now remains to treat the case  $s = 0$ , where if we make no further assumptions than those used in the case  $s > 0$ , the argument breaks down. However if we assume that  $y_1 y_2 < 0$  for all  $(y_1, y_2) \in \text{supp } \widehat{f_1} \times \text{supp } \widehat{f_2}$  (i.e. that the functions  $f_1$  and  $f_2$  on  $\mathbb{R}$  have disjoint angular Fourier support) then it is not hard to see that the change of variables analogous to  $(y_1, y_2) \mapsto (u, v)$  makes sense and the above argument yields an identity corresponding to (1.5) for  $s = 0$ .

## 3. PROOF OF COROLLARIES 1 AND 2

We begin by establishing Corollary 2 and show how to deduce Corollary 1. Before proceeding, we recall the notation

$$\|f\|_{(s)}^4 = \left\| \phi_s(\sqrt{-\Delta})f \right\|_{L^2}^4 - s^2 \left\| (\phi_s(\sqrt{-\Delta}))^{\frac{1}{2}} f \right\|_{L^2}^4.$$

If we set  $d = 5$  and  $f_1 = f_2 = f$  in (1.3), then the right hand side reduces to

$$\begin{aligned} & \int_{\mathbb{R}^{10}} |\widehat{f}(y_1)|^2 |\widehat{f}(y_2)|^2 \phi_s(|y_1|) \phi_s(|y_2|) \frac{(\phi_s(|y_1|) \phi_s(|y_2|) - y_1 \cdot y_2 - s^2)^{\frac{3}{2}}}{(\phi_s(|y_1|) \phi_s(|y_2|) - y_1 \cdot y_2 + s^2)^{\frac{1}{2}}} dy \\ & \leq \int_{\mathbb{R}^{10}} |\widehat{f}(y_1)|^2 |\widehat{f}(y_2)|^2 \phi_s(|y_1|) \phi_s(|y_2|) (\phi_s(|y_1|) \phi_s(|y_2|) - y_1 \cdot y_2 - s^2) dy \\ & = I_1 - I_2, \end{aligned}$$

where

$$I_1 = \int_{\mathbb{R}^{10}} \left( [\phi_s(|y_1|) \phi_s(|y_2|)]^2 - s^2 \phi_s(|y_1|) \phi_s(|y_2|) \right) |\widehat{f}(y_1)|^2 |\widehat{f}(y_2)|^2 dy_1 dy_2,$$

and

$$I_2 = \int_{\mathbb{R}^{10}} |\widehat{f}(y_1)|^2 |\widehat{f}(y_2)|^2 \phi_s(|y_1|) \phi_s(|y_2|) y_1 \cdot y_2 dy_1 dy_2.$$

We can now use the observation of Carneiro in [6],

$$(3.1) \quad \int_{\mathbb{R}^{2d}} f(x) f(y) x \cdot y dx dy \geq 0$$

which holds for any function  $f$ , with equality if  $f$  is radial, to obtain that  $I_2 \geq 0$ . Hence, we have that

$$(3.2) \quad \left\| e^{it\phi_s(\sqrt{-\Delta})} f \right\|_{L_{x,t}^4}^4 \leq (2\pi)^{10} \mathbf{KG}(5) \|f\|_{(s)}^4.$$

Note however that we have used that

$$(3.3) \quad \frac{\phi_s(y_1) \phi_s(y_2) - y_1 \cdot y_2 - s^2}{\phi_s(y_1) \phi_s(y_2) - y_1 \cdot y_2 + s^2} \leq 1,$$

and this inequality is of course pointwise strict, but as with the  $L^\infty$  analysis of the convolution of the measures  $\sigma_s$  in [21] (Corollary 4.3, Lemma 4.4 and Lemma 4.5) we claim that when normalised, the functions  $f_a$  form a maximising sequence for the inequality (3.2), as  $a \rightarrow 0+$ . As a consequence of this and inequality (3.3) we will obtain that the inequality (3.2) is sharp, and that there are no maximisers. We recall that the functions  $f_a$  are defined by

$$\widehat{f}_a(x) = \frac{e^{-a\phi_s(|x|)}}{\phi_s(|x|)},$$

for  $a > 0$ . By Theorem 1, these satisfy inequality (1.3) with equality, and by the observation after inequality (3.1) we also have that  $I_2 = 0$  for such functions.

**Lemma 2.** *Suitably normalised, the functions  $f_a$  form a maximising sequence for the inequality (3.2). That is, we have that*

$$\lim_{a \rightarrow 0+} \frac{\|e^{it\phi_s(\sqrt{-\Delta})} f_a\|_{L_{x,t}^4}^4}{\|f_a\|_{(s)}^4} = (2\pi)^{10} \mathbf{KG}(5).$$

*Proof.* To prove Lemma 2 we modify the approach in [21]. Firstly, we calculate

$$\begin{aligned}
(2\pi)^5 \left\| \phi_s(\sqrt{-\Delta})^\beta f_a \right\|_{L^2}^2 &= \int_{\mathbb{R}^5} \frac{e^{-2a\phi_s(|x|)}}{\phi_s(|x|)^{2-2\beta}} dx \\
&= |\mathbb{S}^4| \int_0^\infty \frac{e^{-2a\sqrt{s^2+r^2}}}{(s^2+r^2)^{1-\beta}} r^4 dr \\
&= |\mathbb{S}^4| \int_s^\infty e^{-2au} (u^2-s^2)^{\frac{3}{2}} u^{2\beta-1} du \\
&= \frac{|\mathbb{S}^4|}{a^{2\beta}} \int_{as}^\infty e^{-2x} \left( \left( \frac{x}{a} \right)^2 - s^2 \right)^{\frac{3}{2}} x^{2\beta-1} dx \\
&= \frac{|\mathbb{S}^4|}{a^{2\beta+3}} \int_{as}^\infty e^{-2x} (x^2 - (as)^2)^{\frac{3}{2}} x^{2\beta-1} dx
\end{aligned}$$

for  $\beta \in \{\frac{1}{2}, 1\}$ , so that

$$(3.4) \quad \lim_{a \rightarrow 0+} a^5 (2\pi)^5 \left\| \phi_s(\sqrt{-\Delta})^\beta f_a \right\|_{L^2}^2 = \begin{cases} \frac{3}{4} |\mathbb{S}^4| & \text{if } \beta = 1, \\ 0 & \text{if } \beta = \frac{1}{2}. \end{cases}$$

We now wish to evaluate

$$\lim_{a \rightarrow 0+} a^{10} \left\| e^{it\phi_s(\sqrt{-\Delta})} f_a \right\|_{L^4}^4.$$

Observe that for these functions  $f_a$  we can write this norm in terms of the convolution of the measure  $\sigma_s$  with itself. Indeed, using Plancherel's theorem and then (2.1) we have

$$\begin{aligned}
\left\| e^{it\phi_s(\sqrt{-\Delta})} f_a \right\|_{L^4}^4 &= \left\| \left( e^{it\phi_s(\sqrt{-\Delta})} f_a \right)^2 \right\|_{L^2}^2 \\
&= \frac{1}{(2\pi)^{18}} \left\| \widetilde{e^{it\phi_s(\sqrt{-\Delta})} f_a} * \widetilde{e^{it\phi_s(\sqrt{-\Delta})} f_a} \right\|_{L^2}^2 \\
&= \frac{1}{(2\pi)^{14}} \left\| \int_{\mathbb{R}^{10}} \frac{e^{-a(\phi_s(|y_1|)\phi_s(|y_2|))}}{\phi_s(|y_1|)\phi_s(|y_2|)} \delta \left( \begin{matrix} \tau - \phi_s(|y_1|) - \phi_s(|y_2|) \\ \xi - y_1 - y_2 \end{matrix} \right) dx \right\|_{L_{\xi, \tau}^2}^2 \\
&= \frac{1}{(2\pi)^{14}} \left\| \int_{\mathbb{R}^{10}} \frac{e^{-a\tau}}{\phi_s(|y_1|)\phi_s(|y_2|)} \delta \left( \begin{matrix} \tau - \phi_s(|y_1|) - \phi_s(|y_2|) \\ \xi - y_1 - y_2 \end{matrix} \right) dx \right\|_{L_{\xi, \tau}^2}^2 \\
&= \frac{1}{(2\pi)^{14}} \left\| e^{-a\tau} \sigma_s * \sigma_s(\tau, \xi) \right\|_{L_{\tau, \xi}^2}^2.
\end{aligned}$$

By Lemma 1, we obtain

$$\begin{aligned}
\left\| e^{it\phi_s(\sqrt{-\Delta})} f_a \right\|_{L^4}^4 &= \frac{|\mathbb{S}^4|^2}{2^6 (2\pi)^{14}} \int_{\mathbb{R}^{5+1}} e^{-2a\tau} \frac{(\tau^2 - |\xi|^2 - 4s^2)^3}{\tau^2 - |\xi|^2} \chi_{\{\tau \geq \sqrt{(2s)^2 + |\xi|^2}\}} d\xi d\tau \\
&= \frac{|\mathbb{S}^4|^2}{2^6 (2\pi)^{14}} \int_{\mathcal{H}_s} e^{-2a\tau} (\tau^2 - |\xi|^2 - 4s^2)^2 \left( 1 - \frac{4s^2}{\tau^2 - |\xi|^2} \right) d\xi d\tau.
\end{aligned}$$

To calculate the integral here, we write

$$\begin{aligned} \left(\tau^2 - |\xi|^2 - 4s^2\right)^2 &= |\xi|^4 + \tau^4 - 2\tau^2 |\xi|^2 - 8s^2\tau^2 + 8s^2 |\xi|^2 + 16s^4 \\ &= \sum_{(j,k) \in \mathcal{T}} c_{j,k} \tau^{2j} |\xi|^{2k}, \end{aligned}$$

where

$$\mathcal{T} = \{(0,0), (1,0), (0,1), (1,1), (0,2), (2,0)\} \subseteq \mathbb{Z} \times \mathbb{Z}.$$

Hence, we obtain that

$$\begin{aligned} &\int_{\mathcal{H}_s} e^{-2a\tau} \left(\tau^2 - |\xi|^2 - 4s^2\right)^2 \left(1 - \frac{4s^2}{\tau^2 - |\xi|^2}\right) d\xi d\tau \\ &= \sum_{(j,k) \in \mathcal{T}} c_{j,k} \int_{2s}^{\infty} e^{-2a\tau} \tau^{2j} \int_{|\xi| \leq \sqrt{\tau^2 - (2s)^2}} |\xi|^{2k} \left(1 - \frac{4s^2}{\tau^2 - |\xi|^2}\right) d\xi d\tau \\ &= |\mathbb{S}^4| \sum_{(j,k) \in \mathcal{T}} c_{j,k} (I_{j,k} - 4s^2 \Pi_{j,k}), \end{aligned}$$

where we define

$$I_{j,k} = \int_{2s}^{\infty} e^{-2a\tau} \tau^{2j} \int_0^{\sqrt{\tau^2 - (2s)^2}} r^{2(k+2)} dr d\tau,$$

and

$$\Pi_{j,k} = \int_{2s}^{\infty} e^{-2a\tau} \tau^{2j} \int_0^{\sqrt{\tau^2 - (2s)^2}} \frac{r^{2(k+2)}}{\tau^2 - r^2} dr d\tau.$$

**Claim 1.** *We have that*

$$\lim_{a \rightarrow 0+} a^{10} \Pi_{j,k} = \lim_{a \rightarrow 0+} a^{10} \Pi_{j+1,k-1},$$

provided that  $j+k < 3$ ,  $k > -\frac{3}{2}$  and  $j \geq 0$ , and

$$\lim_{a \rightarrow 0+} a^{10} \Pi_{j,-1} = 0,$$

for  $j \in \{1, 2, 3\}$ .

Assuming the claim to be true for the moment, it then follows that  $\Pi_{j,k} = 0$  for each pair  $(j,k) \in \mathcal{T}$ , and hence

$$\lim_{a \rightarrow 0+} a^{10} \|e^{-a\tau} \sigma_s * \sigma_s(\tau, \xi)\|_{L_{\tau, \xi}^2}^2 = \frac{|\mathbb{S}^4|^3}{2^6} \sum_{(j,k) \in \mathcal{T}} c_{j,k} I_{j,k},$$

and we note that we can evaluate  $I_{j,k}$  directly, as

$$\begin{aligned} a^{10} I_{j,k} &= \frac{a^{10}}{2k+5} \int_{2s}^{\infty} e^{-2a\tau} \tau^{2j} (\tau^2 - (2s)^2)^{\frac{2k+5}{2}} d\tau \\ &= \frac{1}{2k+5} a^{4-2(j+k)} \int_{2as}^{\infty} e^{-2x} x^{2j} (x^2 - (2as)^2)^{\frac{2k+5}{2}} dx. \end{aligned}$$

Hence, since the latter integral converges for any  $a > 0$  we have

$$\lim_{a \rightarrow 0+} a^{10} I_{j,k} = \begin{cases} \frac{1}{2k+5} \int_0^{\infty} e^{-2x} x^9 dx & \text{if } j+k=2, \\ 0 & \text{if } j+k < 2. \end{cases}$$

In all, since

$$\int_0^\infty x^\ell e^{-2x} dx = \frac{\ell!}{2^{\ell+1}}$$

we obtain that

$$\lim_{a \rightarrow 0+} a^{10} \left\| e^{it\phi_s(\sqrt{-\Delta})} f_a \right\|_{L^4}^4 = \frac{|\mathbb{S}^4|^3}{(2\pi)^{14}} \left( \frac{1}{5} + \frac{1}{9} - \frac{2}{7} \right) \frac{9!}{2^{16}},$$

and

$$\lim_{a \rightarrow 0+} a^{10} (2\pi)^{10} \|f_a\|_{(s)}^4 = \frac{9}{16} |\mathbb{S}^4|^2.$$

Hence

$$\lim_{a \rightarrow 0+} \frac{\left\| e^{it\phi_s(\sqrt{-\Delta})} f_a \right\|_{L^4}^4}{(2\pi)^{10} \|f_a\|_{(s)}^4} = \mathbf{KG}(5),$$

as claimed, and therefore the constant  $(2\pi)^{10} \mathbf{KG}(5)$  is best possible for the inequality (1.12). We also remark at this point that the constant  $(2\pi)^{10} \mathbf{KG}(5)$  is also sharp for the inequality (1.16); as discussed in Section 1 this follows from the sequence  $(g_a)_{a>0}$  defined by (1.13), since by (3.4) we have  $a^5 \left\| \phi_s(\sqrt{-\Delta})^{\frac{1}{2}} f_a \right\|_{L^2}^2 \rightarrow 0$  as  $a \rightarrow 0+$ . It now remains to prove Claim 1.

*Proof of Claim 1.* For the first part, we note that

$$\begin{aligned} \int_0^{\sqrt{\tau^2 - (2s)^2}} \frac{r^{2(k+2)}}{\tau^2 - r^2} dr &= \int_0^{\sqrt{\tau^2 - (2s)^2}} r^{2(k+1)} \left( \frac{\tau^2}{\tau^2 - r^2} - 1 \right) dr \\ &= \tau^2 \int_0^{\sqrt{\tau^2 - (2s)^2}} \frac{r^{2(k+1)}}{\tau^2 - r^2} dr - \frac{1}{2k+3} (\tau^2 - (2s)^2)^{k+\frac{3}{2}}. \end{aligned}$$

But then,

$$\begin{aligned} a^{10} \int_{2s}^\infty e^{-2a\tau} \tau^{2j} (\tau^2 - (2s)^2)^{k+\frac{3}{2}} d\tau &= a^{6-2(j+k)} \int_{2as}^\infty e^{-2x} x^{2j} (x^2 - (2as)^2)^{k+\frac{3}{2}} dx \\ &\rightarrow 0 \end{aligned}$$

as  $a \rightarrow 0+$  since  $j+k < 3$  and  $j \geq 0$ . Hence,

$$\begin{aligned} \lim_{a \rightarrow 0+} a^{10} \Pi_{j,k} &= \lim_{a \rightarrow 0+} a^{10} \int_{2s}^\infty e^{-2a\tau} \tau^{2(j+1)} \int_0^{\sqrt{\tau^2 - (2s)^2}} \frac{r^{2(k+1)}}{\tau^2 - r^2} dr d\tau \\ &= \lim_{a \rightarrow 0+} a^{10} \Pi_{j+1,k-1}. \end{aligned}$$

For the second part, by a simple change of variables we can calculate  $\Pi_{j,-1}$  directly, as in [21]. We have

$$\begin{aligned} a^{10} \Pi_{j,-1} &= a^{8-2j} \int_{2as}^\infty e^{-2x} x^{2j+1} \log \left( \frac{x + \sqrt{x^2 + (2as)}}{2as} \right) dx \\ &= a^{8-2j} \int_{2as}^\infty e^{-2x} x^{2j+1} \log \left( x + \sqrt{x^2 + (2as)} \right) dx \\ &\quad - a^{8-2j} \log(2as) \int_{2as}^\infty e^{-2x} x^{2j+1} dx \\ &\rightarrow 0 \end{aligned}$$

as  $a \rightarrow 0+$ , since  $8-2j > 0$  and  $2j+1 > 0$ . □

We conclude the section by showing how Corollary 1 is deduced from Corollary 2; to do this we follow the approach of Foschi in [13]. Suppose  $u$  solves

$$(3.5) \quad \partial_{tt}u - \Delta u + u = 0,$$

where  $u = u(x, t)$  is a function defined on  $\mathbb{R}^{5+1}$ . Then we can write  $u = u_+ + u_-$ , where

$$(3.6) \quad u_+ = e^{it\phi_1(\sqrt{-\Delta})}f_+, \quad u_- = e^{-it\phi_1(\sqrt{-\Delta})}f_-,$$

for functions  $f_+$  and  $f_-$  defined using the initial data by

$$u(0) = f_+ + f_-, \quad \partial_t u(0) = i\phi_1(\sqrt{-\Delta})(f_+ - f_-).$$

Then

$$\|u\|_{L^4}^4 = \|u_+ + u_-\|_{L^4}^4 = \|u_+^2 + u_-^2 + 2u_+u_-\|_{L^2}^2.$$

We claim that the supports of the space-time Fourier transforms of the functions  $u_+^2$ ,  $u_-^2$  and  $u_+u_-$  are pairwise disjoint. We have already seen that

$$\text{supp } \widetilde{u_+} \subseteq \left\{ (\xi, \tau) \in \mathbb{R}^{d+1} : \tau \geq \sqrt{4 + |\xi|^2} \right\},$$

and by an identical argument we have that

$$\text{supp } \widetilde{u_-} \subseteq \left\{ (\xi, \tau) \in \mathbb{R}^{d+1} : \tau \leq -\sqrt{4 + |\xi|^2} \right\}.$$

It remains to show that

$$\text{supp } \widetilde{u_+u_-} \subseteq \left\{ (\xi, \tau) \in \mathbb{R}^{d+1} : |\tau| \leq \sqrt{4 + |\xi|^2} \right\}.$$

We note that this was shown in [21], we include it here for completeness. To see this, note that analogously to (2.1) we will have, for  $(\xi, \tau) \in \mathbb{R}^{d+1}$ ,

$$\widetilde{u_+u_-}(\xi, \tau) = \frac{1}{(2\pi)^{d-1}} \int_{\mathbb{R}^{2d}} \frac{\widehat{F}(y)}{(1 + |y_1|^2)^{\frac{1}{4}}(1 + |y_2|^2)^{\frac{1}{4}}} \delta \left( \begin{array}{c} \tau - \phi_1(|y_1|) + \phi_1(|y_2|) \\ \xi - y_1 - y_2 \end{array} \right) dy.$$

Set  $\xi = y_1 + y_2$  and  $\tau = \phi_1(|y_1|) - \phi_1(|y_2|)$ , then we have

$$|\xi|^2 = |y_1|^2 + |y_2|^2 + 2y_1 \cdot y_2,$$

and,

$$\tau^2 = 2s^2 + |y_1|^2 + |y_2|^2 - 2\phi_1(|y_1|)\phi_1(|y_2|),$$

so that

$$\tau^2 - |\xi|^2 = 2 - 2\phi_1(|y_1|)\phi_1(|y_2|) - 2y_1 \cdot y_2 \leq 4,$$

where the final inequality follows from (2.2). Hence,

$$\|u\|_{L^4}^4 = \|u_+^2\|_2^2 + \|u_-^2\|_2^2 + 4\|u_+u_-\|_2^2.$$

Combining the above equality with the sharp polynomial inequality for non-negative real numbers  $X$  and  $Y$

$$X^2 + Y^2 + 4XY \leq \frac{3}{2}(X + Y)^2$$

with equality if and only if  $X = Y$ , we obtain the following sharp inequality in terms of the one-sided propagators:

$$\|u\|_{L^4}^4 \leq \frac{3}{2} \left( \|u_+\|_{L^4}^2 + \|u_-\|_{L^4}^2 \right)^2.$$

Using the inequality (1.16), it follows that

$$\|u\|_{L^4}^4 \leq \frac{1}{16\pi^2} \left( \|f_+\|_{H^1}^2 + \|f_-\|_{H^1}^2 \right)^2.$$

But then, by the definition of  $f_+$  and  $f_-$  and the parallelogram law, the right hand side equals

$$\frac{1}{16\pi^2} \left( \frac{1}{2} \|u(0)\|_{H^1}^2 + \frac{1}{2} \left\| \phi_1(\sqrt{-\Delta})^{-1} \partial_t u(0) \right\|_{H^1}^2 \right)^2 = \frac{1}{64\pi^2} \left( \|u(0)\|_{H^1}^2 + \|\partial_t u(0)\|_{L^2}^2 \right)^2,$$

which completes the proof of Corollary 1.  $\square$

#### 4. PROOF OF PROPOSITION 1

In this section, it will be convenient to abuse notation slightly and think of  $\phi_s(x)$  as a function on  $\mathbb{R}^5$  by identifying with  $\phi_s(|x|)$ . If we let

$$\|g \otimes g\|_{(\tau, \xi)}^2 = \int_{\mathbb{R}^{10}} |g(y_1)|^2 |g(y_2)|^2 \delta \left( \begin{array}{c} \tau - \phi_s(|y_1|) - \phi_s(|y_2|) \\ \xi - y_1 - y_2 \end{array} \right) dy_1 dy_2,$$

by the proof of the bilinear inequality of Theorem 1 and by Lemma 1, we have that

$$\begin{aligned} & \left\| e^{it\phi_s(\sqrt{-\Delta})} g_n \right\|_{L^4(\mathbb{R}^6)}^4 \\ & \leq \frac{1}{24\pi^2} \int_{\mathcal{H}_s} \left\| \phi_s^{\frac{1}{2}} \widehat{g_n} \otimes \phi_s^{\frac{1}{2}} \widehat{g_n} \right\|_{(\tau, \xi)}^2 \frac{(\tau^2 - |\xi|^2 - 4s^2)^{\frac{3}{2}}}{(\tau^2 - |\xi|^2)^{\frac{1}{2}}} d\xi d\tau \\ & = \frac{1}{24\pi^2} \int_{\mathcal{H}_s} \left\| \phi_s^{\frac{1}{2}} \widehat{g_n} \otimes \phi_s^{\frac{1}{2}} \widehat{g_n} \right\|_{(\tau, \xi)}^2 \left( \tau^2 - |\xi|^2 \right) \left( 1 - \frac{4s^2}{\tau^2 - |\xi|^2} \right)^{\frac{3}{2}} d\tau d\xi \\ & = \frac{1}{24\pi^2} \left( \|g_n\|_{(s)}^4 - \mathcal{I}_n - \mathcal{J}_n \right) \\ & \leq \frac{1}{24\pi^2} (1 - \mathcal{I}_n - \mathcal{J}_n), \end{aligned}$$

where

$$\mathcal{I}_n = \int_{\mathbb{R}^{10}} |\widehat{g_n}(y_1)|^2 |\widehat{g_n}(y_2)|^2 \phi_s(|y_1|) \phi_s(|y_2|) y_1 \cdot y_2 dy_1 dy_2 \geq 0,$$

and

$$\mathcal{J}_n = \int_{\mathcal{H}_s} \left\| \phi_s^{\frac{1}{2}} \widehat{g_n} \otimes \phi_s^{\frac{1}{2}} \widehat{g_n} \right\|_{(\tau, \xi)}^2 \left( \tau^2 - |\xi|^2 \right) \left( 1 - \left( 1 - \frac{4s^2}{\tau^2 - |\xi|^2} \right)^{\frac{3}{2}} \right) d\tau d\xi \geq 0.$$

But then since  $(g_n)_{n \geq 1}$  is a maximising sequence for inequality (1.12), it follows that  $\mathcal{I}_n, \mathcal{J}_n \rightarrow 0$  as  $n \rightarrow \infty$ . Firstly, since  $0 < 1 - \frac{4s^2}{\tau^2 - |\xi|^2} < 1$ , we obtain

$$\left( \int_{\mathbb{R}^5} |\widehat{g_n}(y_1)|^2 \phi_s(|y_1|) dy_1 \right)^2 < C \mathcal{J}_n,$$

for some positive constant  $C$ , and hence

$$\int_{\mathbb{R}^5} |\widehat{g_n}(y_1)|^2 \phi_s(|y_1|) dy_1 \rightarrow 0$$

as  $n \rightarrow \infty$ .

Now, to prove (1.18), using the fact that on the delta measures we have that

$$(4.1) \quad \phi_s(|y_1|)\phi_s(|y_2|) - y_1 \cdot y_2 + s^2 = \frac{1}{2} \left( \tau^2 - |\xi|^2 \right),$$

if  $y_1, y_2 \in B(0, R)$  it is easy to see that for such  $\tau, \xi$ ,

$$\tau^2 - |\xi|^2 \leq 2(R^2 + s^2).$$

Thus,

$$\begin{aligned} & \int_{\mathcal{H}^s} \int_{B(0, R)} |\widehat{g_n}(y_1)|^2 |\widehat{g_n}(y_2)|^2 \phi_s(|y_1|)\phi_s(|y_2|) \left( \tau^2 - |\xi|^2 \right) \\ & \quad \times \delta \left( \begin{matrix} \tau - \phi_s(|y_1|) - \phi_s(|y_2|) \\ \xi - y_1 - y_2 \end{matrix} \right) dy d\xi d\tau \\ & \leq 2(R^2 + s^2) \left( \int_{\mathbb{R}^5} |\widehat{g_n}(y_1)|^2 \phi_s(|y_1|) dy_1 \right)^2 \rightarrow 0 \end{aligned}$$

as  $n \rightarrow \infty$ . Using (4.1) we obtain

$$\begin{aligned} & \left( \int_{B(0, R)} |\widehat{g_n}(y_1)|^2 \phi_s(|y_1|)^2 dy_1 \right)^2 \\ & \leq \int_{B(0, R)} |\widehat{g_n}(y_1)|^2 |\widehat{g_n}(y_2)|^2 \left( \phi_s(|y_1|)^2 \phi_s(|y_2|)^2 + s^2 \phi_s(|y_1|)\phi_s(|y_2|) \right) dy_1 dy_2 \\ & = \int_{B(0, R)} |\widehat{g_n}(y_1)|^2 |\widehat{g_n}(y_2)|^2 \phi_s(|y_1|)\phi_s(|y_2|) y_1 \cdot y_2 dy_1 dy_2 \\ & \quad + \frac{1}{2} \int_{\mathcal{H}^s} \int_{B(0, R)} |\widehat{g_n}(y_1)|^2 |\widehat{g_n}(y_2)|^2 \phi_s(|y_1|)\phi_s(|y_2|) \left( \tau^2 - |\xi|^2 \right) \\ & \quad \times \delta \left( \begin{matrix} \tau - \phi_s(|y_1|) - \phi_s(|y_2|) \\ \xi - y_1 - y_2 \end{matrix} \right) dy d\xi d\tau \\ & \leq \mathcal{I}_n + (R^2 + s^2) \left( \int_{\mathbb{R}^5} |\widehat{g_n}(y_1)|^2 \phi_s(|y_1|) dy_1 \right)^2 \rightarrow 0 \end{aligned}$$

as  $n \rightarrow \infty$ . But then if  $\varepsilon, R$  are given then we can choose  $N$ , as desired.

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SCHOOL OF MATHEMATICS, THE WATSON BUILDING, THE UNIVERSITY OF BIRMINGHAM, EDGBASTON, BIRMINGHAM, B15 2TT, ENGLAND

*E-mail address:* jeavonsc@maths.bham.ac.uk